Pang and Ychussie *Journal of Inequalities and Applications* (2015) 2015:108
DOI 10.1186/s13660-015-0621-8

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Matsaev type inequalities on smooth cones

Sheng Pang[1] and Beatriz Ychussie[2*]

*Correspondence:
ychussie.b@gmail.com
[2]Mathematics Institute, Roskilde University, Roskilde, 4000, Denmark
Full list of author information is available at the end of the article

Abstract

Our aim in this paper is to obtain Matsaev type inequalities about harmonic functions on smooth cones, which generalize the results obtained by Xu, Yang and Zhao in a half space.

MSC: 31B05; 31B10

Keywords: Matsaev type inequality; harmonic function; cone

1 Introduction and results

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the set of all real numbers and the set of all positive real numbers, respectively. We denote by $\mathbf{R}^n$ ($n \geq 2$) the n-dimensional Euclidean space. A point in $\mathbf{R}^n$ is denoted by $P = (X, x_n)$, $X = (x_1, x_2, \ldots, x_{n-1})$. The Euclidean distance between two points P and Q in $\mathbf{R}^n$ is denoted by $|P - Q|$. Also $|P - O|$ with the origin O of $\mathbf{R}^n$ is simply denoted by $|P|$. The boundary and the closure of a set S in $\mathbf{R}^n$ are denoted by ∂S and $\overline{S}$, respectively.

We introduce a system of spherical coordinates (r, Θ), $\Theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})$, in $\mathbf{R}^n$ which are related to Cartesian coordinates $(x_1, x_2, \ldots, x_{n-1}, x_n)$ by $x_n = r\cos\theta_1$.

The unit sphere and the upper half unit sphere in $\mathbf{R}^n$ are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}^{n-1}_+$, respectively. For simplicity, a point $(1, \Theta)$ on $\mathbf{S}^{n-1}$ and the set $\{\Theta; (1, \Theta) \in \Omega\}$ for a set Ω, $\Omega \subset \mathbf{S}^{n-1}$, are often identified with Θ and Ω, respectively. For two sets $\Xi \subset \mathbf{R}_+$ and $\Omega \subset \mathbf{S}^{n-1}$, the set $\{(r, \Theta) \in \mathbf{R}^n; r \in \Xi, (1, \Theta) \in \Omega\}$ in $\mathbf{R}^n$ is simply denoted by $\Xi \times \Omega$. In particular, the half space $\mathbf{R}_+ \times \mathbf{S}^{n-1}_+ = \{(X, x_n) \in \mathbf{R}^n; x_n > 0\}$ will be denoted by T_n.

For $P \in \mathbf{R}^n$ and $r > 0$, let $B(P, r)$ denote the open ball with center at P and radius r in $\mathbf{R}^n$. $S_r = \partial B(O, r)$. By $C_n(\Omega)$, we denote the set $\mathbf{R}_+ \times \Omega$ in $\mathbf{R}^n$ with the domain Ω on $\mathbf{S}^{n-1}$. We call it a cone. Then T_n is a special cone obtained by putting $\Omega = \mathbf{S}^{n-1}_+$. We denote the sets $I \times \Omega$ and $I \times \partial\Omega$ with an interval on $\mathbf{R}$ by $C_n(\Omega; I)$ and $S_n(\Omega; I)$. By $S_n(\Omega; r)$ we denote $C_n(\Omega) \cap S_r$. By $S_n(\Omega)$ we denote $S_n(\Omega; (0, +\infty))$ which is $\partial C_n(\Omega) - \{O\}$.

We use the standard notations $u^+ = \max\{u, 0\}$ and $u^- = -\min\{u, 0\}$. Further, we denote by w_n the surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of $\mathbf{S}^{n-1}$, by $\partial/\partial n_Q$ denotes the differentiation at Q along the inward normal into $C_n(\Omega)$, by dS_r the $(n-1)$-dimensional volume elements induced by the Euclidean metric on S_r and by dw the elements of the Euclidean volume in $\mathbf{R}^n$.

Let Ω be a domain on $\mathbf{S}^{n-1}$ with smooth boundary. Consider the Dirichlet problem

$$(\Lambda_n + \lambda)\varphi = 0 \quad \text{on } \Omega,$$

$$\varphi = 0 \quad \text{on } \partial\Omega,$$

where Λ_n is the spherical part of the Laplace operator Δ_n,

$$\Delta_n = \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{\Lambda_n}{r^2}.$$

We denote the least positive eigenvalue of this boundary value problem by λ and the normalized positive eigenfunction corresponding to λ by $\varphi(\Theta)$, $\int_\Omega \varphi^2(\Theta)\,dS_1 = 1$. In order to ensure the existence of λ and a smooth $\varphi(\Theta)$. We put a rather strong assumption on Ω: if $n \geq 3$, then Ω is a $C^{2,\alpha}$-domain ($0 < \alpha < 1$) on $\mathbf{S}^{n-1}$ surrounded by a finite number of mutually disjoint closed hypersurfaces (*e.g.* see [1], pp.88-89, for the definition of $C^{2,\alpha}$-domain). Then $\varphi \in C^2(\overline{\Omega})$ and $\partial\varphi/\partial n > 0$ on $\partial\Omega$ (here and below, $\partial/\partial n$ denotes differentiation along the interior normal).

We note that each function

$$r^{\aleph^\pm}\varphi(\Theta)$$

is harmonic in $C_n(\Omega)$, belongs to the class $C^2(C_n(\Omega)\backslash\{O\})$ and vanishes on $S_n(\Omega)$, where

$$2\aleph^\pm = -n + 2 \pm \sqrt{(n-2)^2 + 4\lambda}.$$

In the sequel, for the sake of brevity, we shall write χ instead of $\aleph^+ - \aleph^-$. If $\Omega = \mathbf{S}_+^{n-1}$, then $\aleph^+ = 1$, $\aleph^- = 1 - n$, and $\varphi(\Theta) = (2nw_n^{-1})^{1/2}\cos\theta_1$.

Let $G_\Omega(P,Q)$ ($P = (r,\Theta)$, $Q = (t,\Phi) \in C_n(\Omega)$) be the Green function of $C_n(\Omega)$. Then the ordinary Poisson kernel relative to $C_n(\Omega)$ is defined by

$$\mathcal{PI}_\Omega(P,Q) = \frac{1}{c_n}\frac{\partial}{\partial n_Q}G_\Omega(P,Q),$$

where $Q \in S_n(\Omega)$ and

$$c_n = \begin{cases} 2\pi & \text{if } n = 2, \\ (n-2)w_n & \text{if } n \geq 3. \end{cases}$$

The estimate we deal with has a long history which can be traced back to Matsaev's estimate of harmonic functions from below (see, for example, Levin [2], p.209).

Theorem A *Let A_1 be a constant, $u(z)$ ($|z| = R$) be harmonic on T_2 and continuous on ∂T_2. Suppose that*

$$u(z) \leq A_1 R^\rho, \quad z \in T_2, R > 1, \rho > 1$$

and

$$\left|u(z)\right| \leq A_1, \quad R \leq 1, z \in \overline{T}_2.$$

Then

$$u(z) \geq -A_1 A_2\left(1 + R^\rho\right)\sin^{-1}\alpha,$$

where $z = Re^{i\alpha} \in T_2$ and A_2 is a constant independent of A_1, R, α, and the function $u(z)$.

Recently, Xu *et al.* [3–5] considered Theorem A in the n-dimensional ($n \geq 2$) case and obtained the following result.

Theorem B *Let A_3 be a constant, $u(P)$ ($|P| = R$) be harmonic on T_n and continuous on $\overline{T}_n$. If*

$$u(P) \leq A_3 R^\rho, \quad P \in T_n, R > 1, \rho > n-1 \tag{1.1}$$

and

$$\bigl|u(P)\bigr| \leq A_3, \quad R \leq 1, P \in \overline{T}_n, \tag{1.2}$$

then

$$u(P) \geq -A_3 A_4 \bigl(1 + R^\rho\bigr) \cos^{1-n} \theta_1,$$

where $P \in T_n$ and A_4 is a constant independent of A_3, R, θ_1, and the function $u(P)$.

Now we have the following.

Theorem 1 *Let K be a constant, $u(P)$ ($P = (R, \Theta)$) be harmonic on $C_n(\Omega)$ and continuous on $\overline{C_n(\Omega)}$. If*

$$u(P) \leq K R^{\rho(R)}, \quad P = (R, \Theta) \in C_n\bigl(\Omega; (1, \infty)\bigr), \rho(R) > \aleph^+ \tag{1.3}$$

and

$$u(P) \geq -K, \quad R \leq 1, P = (R, \Theta) \in \overline{C_n(\Omega)}, \tag{1.4}$$

then

$$u(P) \geq -KM\left(1 + \left(\frac{N+1}{N} R\right)^{\rho(\frac{N+1}{N} R)}\right) \varphi^{1-n} \theta,$$

where $P \in C_n(\Omega)$, N (≥ 1) is a sufficiently large number, $\rho(R)$ is nondecreasing in $[1, +\infty)$ and M is a constant independent of K, R, $\varphi(\theta)$, and the function $u(P)$.

By taking $\rho(R) \equiv \rho$, we obtain the following corollary, which generalizes Theorem B to the conical case.

Corollary *Let K be a constant, $u(P)$ ($P = (R, \Theta)$) be harmonic on $C_n(\Omega)$ and continuous on $\overline{C_n(\Omega)}$. If*

$$u(P) \leq K R^\rho, \quad P = (R, \Theta) \in C_n\bigl(\Omega; (1, \infty)\bigr), \rho > \aleph^+$$

and

$$u(P) \geq -K, \quad R \leq 1, P = (R, \Theta) \in \overline{C_n(\Omega)},$$

then

$$u(P) \geq -KM\bigl(1 + R^{\rho}\bigr)\varphi^{1-n}\theta,$$

where $P \in C_n(\Omega)$, M *is a constant independent of* K, R, $\varphi(\theta)$, *and the function* $u(P)$.

Remark From the corollary, we know that conditions (1.1) and (1.2) may be replaced with the weaker conditions

$$u(P) \leq A_3 R^{\rho}, \quad P \in T_n, R > 1, \rho > 1$$

and

$$u(P) \geq -A_3, \quad R \leq 1, P \in \overline{T}_n,$$

respectively.

2 Lemmas

Throughout this paper, let M denote various constants independent of the variables in question, which may be different from line to line.

Carleman's formula (see [6]) connects the modulus and the zeros of a function analytic in a complex plane (see, for example, [7], p.224). I Miyamoto and H Yoshida generalized it to subharmonic functions in an n-dimensional cone (see [8, 9]).

Lemma 1 *If* $R > 1$ *and* $u(t,\Phi)$ *is a subharmonic function on a domain containing* $C_n(\Omega;(1,R))$, *then*

$$\begin{aligned} &\int_{C_n(\Omega;(1,R))} \left(\frac{1}{t^{-\aleph^-}} - \frac{t^{\aleph^+}}{R^{\chi}}\right)\varphi \Delta u \, dw \\ &\quad = \chi \int_{S_n(\Omega;R)} \frac{u\varphi}{R^{1-\aleph^-}} \, dS_R + \int_{S_n(\Omega;(1,R))} u\left(\frac{1}{t^{-\aleph^-}} - \frac{t^{\aleph^+}}{R^{\chi}}\right)\frac{\partial \varphi}{\partial n} \, d\sigma_Q + d_1 + \frac{d_2}{R^{\chi}}, \end{aligned}$$

where

$$d_1 = \int_{S_n(\Omega;1)} \aleph^- u\varphi - \varphi \frac{\partial u}{\partial n} \, dS_1 \quad \text{and} \quad d_2 = \int_{S_n(\Omega;1)} \varphi \frac{\partial u}{\partial n} - \aleph^+ u\varphi \, dS_1.$$

Lemma 2 (see [8, 9])

$$\mathcal{PI}_{\Omega}(P,Q) \leq M r^{\aleph^-} t^{\aleph^+ - 1} \varphi(\Theta) \frac{\partial \varphi(\Phi)}{\partial n_{\Phi}} \tag{2.1}$$

for any $P = (r,\Theta) \in C_n(\Omega)$ *and any* $Q = (t,\Phi) \in S_n(\Omega)$ *satisfying* $0 < \frac{t}{r} \leq \frac{4}{5}$,

$$\mathcal{PI}_{\Omega}(P,Q) \leq M \frac{\varphi(\Theta)}{t^{n-1}} \frac{\partial \varphi(\Phi)}{\partial n_{\Phi}} + M \frac{r\varphi(\Theta)}{|P-Q|^{n}} \frac{\partial \varphi(\Phi)}{\partial n_{\Phi}} \tag{2.2}$$

for any $P = (r,\Theta) \in C_n(\Omega)$ *and any* $Q = (t,\Phi) \in S_n(\Omega;(\frac{4}{5}r, \frac{5}{4}r))$.

Let $G_{\Omega,R}(P,Q)$ be the Green function of $C_n(\Omega,(0,R))$. Then

$$\frac{\partial G_{\Omega,R}(P,Q)}{\partial R} \le Mr^{\aleph^+}R^{\aleph^- -1}\varphi(\Theta)\varphi(\Phi), \tag{2.3}$$

where $P=(r,\Theta)\in C_n(\Omega)$ and $Q=(R,\Phi)\in S_n(\Omega;R)$.

3 Proof of Theorem 1

Lemma 1 applied to $u=u^+-u^-$ gives

$$\begin{aligned}&\chi\int_{S_n(\Omega;R)}\frac{u^+\varphi}{R^{1-\aleph^-}}\,dS_R+\int_{S_n(\Omega;(1,R))}u^+\left(\frac{1}{t^{-\aleph^-}}-\frac{t^{\aleph^+}}{R^\chi}\right)\frac{\partial\varphi}{\partial n}\,d\sigma_Q+d_1+\frac{d_2}{R^\chi}\\&\quad=\chi\int_{S_n(\Omega;R)}\frac{u^-\varphi}{R^{1-\aleph^-}}\,dS_R+\int_{S_n(\Omega;(1,R))}u^-\left(\frac{1}{t^{-\aleph^-}}-\frac{t^{\aleph^+}}{R^\chi}\right)\frac{\partial\varphi}{\partial n}\,d\sigma_Q.\end{aligned}\tag{3.1}$$

It immediately follows from (1.3) that

$$\chi\int_{S_n(\Omega;R)}\frac{u^+\varphi}{R^{1-\aleph^-}}\,dS_R\le MKR^{\rho(R)-\aleph^+}\tag{3.2}$$

and

$$\begin{aligned}&\int_{S_n(\Omega;(1,R))}u^+\left(\frac{1}{t^{-\aleph^-}}-\frac{t^{\aleph^+}}{R^\chi}\right)\frac{\partial\varphi}{\partial n}\,d\sigma_Q\\&\quad\le\int_{S_n(\Omega;(1,R))}Kt^{\rho(t)+\aleph^+}\left(\frac{1}{t^\chi}-\frac{1}{R^\chi}\right)\frac{\partial\varphi}{\partial n}\,d\sigma_Q\\&\quad\le MK\int_1^R\left(r^{\rho(r)-\aleph^+-1}-\frac{r^{\rho(r)-\aleph^--1}}{R^\chi}\right)\frac{\partial\varphi}{\partial n}\,dr\\&\quad\le MK\int_1^R r^{\rho(R)-\aleph^+-1}\,dr\\&\quad\le\frac{MK}{\rho(R)-\aleph^+}R^{\rho(R)-\aleph^+}\\&\quad\le MKR^{\rho(R)-\aleph^+}.\end{aligned}\tag{3.3}$$

Notice that

$$d_1+\frac{d_2}{R^\chi}\le MKR^{\rho(R)-\aleph^+}.\tag{3.4}$$

Hence from (3.1), (3.2), (3.3), and (3.4) we have

$$\chi\int_{S_n(\Omega;R)}\frac{u^-\varphi}{R^{1-\aleph^-}}\,dS_R\le MKR^{\rho(R)-\aleph^+}\tag{3.5}$$

and

$$\int_{S_n(\Omega;(1,R))}u^-\left(\frac{1}{t^{-\aleph^-}}-\frac{t^{\aleph^+}}{R^\chi}\right)\frac{\partial\varphi}{\partial n}\,d\sigma_Q\le MKR^{\rho(R)-\aleph^+}.\tag{3.6}$$

Equation (3.6) gives

$$\begin{aligned}
&\int_{S_n(\Omega;(1,R))} u^- t^{\aleph^-} \frac{\partial \varphi}{\partial n}\, d\sigma_Q \\
&\quad \le \frac{(N+1)^\chi}{(N+1)^\chi - N^\chi} \int_{S_n(\Omega;(1,\frac{N+1}{N}R))} u^- \left(\frac{1}{t^{-\aleph^-}} - \frac{t^{\aleph^+}}{(\frac{N+1}{N}R)^\chi} \right) \frac{\partial \varphi}{\partial n}\, d\sigma_Q \\
&\quad \le \frac{(N+1)^\chi}{(N+1)^\chi - N^\chi} MK \left(\frac{N+1}{N} R \right)^{\rho(\frac{N+1}{N}R) - \aleph^+} \\
&\quad \le MK \left(\frac{N+1}{N} R \right)^{\rho(\frac{N+1}{N}R) - \aleph^+}.
\end{aligned}$$

Thus

$$\int_{S_n(\Omega;(1,R))} u^- t^{\aleph^-} \frac{\partial \varphi}{\partial n}\, d\sigma_Q \le MK \left(\frac{N+1}{N} R \right)^{\rho(\frac{N+1}{N}R) - \aleph^+}. \tag{3.7}$$

By the Riesz decomposition theorem (see [7]), for any $P = (r, \Theta) \in C_n(\Omega; (0, R))$ we have

$$\begin{aligned}
-u(P) &= \int_{S_n(\Omega;(0,R))} \mathcal{PI}_\Omega(P,Q) - u(Q)\, d\sigma_Q \\
&\quad + \int_{S_n(\Omega;R)} \frac{\partial G_{\Omega,R}(P,Q)}{\partial R} - u(Q)\, dS_R.
\end{aligned} \tag{3.8}$$

Now we distinguish three cases.
Case 1. $P = (r, \Theta) \in C_n(\Omega; (\frac{5}{4}, \infty))$ and $R = \frac{5}{4} r$.
Since $-u(x) \le u^-(x)$, we obtain

$$-u(P) = \sum_{i=1}^{4} I_i(P) \tag{3.9}$$

from (3.8), where

$$\begin{aligned}
I_1(P) &= \int_{S_n(\Omega;(0,1])} \mathcal{PI}_\Omega(P,Q) - u(Q)\, d\sigma_Q, \\
I_2(P) &= \int_{S_n(\Omega;(1,\frac{4}{5}r])} \mathcal{PI}_\Omega(P,Q) - u(Q)\, d\sigma_Q, \\
I_3(P) &= \int_{S_n(\Omega;(\frac{4}{5}r,R))} \mathcal{PI}_\Omega(P,Q) - u(Q)\, d\sigma_Q \quad \text{and} \\
I_4(P) &= \int_{S_n(\Omega;R)} \mathcal{PI}_\Omega(P,Q) - u(Q)\, d\sigma_Q.
\end{aligned}$$

Then from (2.1) and (3.7) we have

$$I_1(P) \le MK\varphi(\Theta) \tag{3.10}$$

and

$$I_2(P) \le r^{\aleph^-}\varphi(\Theta)\left(\frac{4}{5}r\right)^{\chi-1}\int_{S_n(\Omega;(1,\frac{4}{5}r])} -u(Q)t^{\aleph^-}\frac{\partial\varphi(\Phi)}{\partial n_\Phi}\,d\sigma_Q$$

$$\le MK\left(\frac{N+1}{N}R\right)^{\rho(\frac{N+1}{N}R)}\varphi(\Theta). \tag{3.11}$$

By (2.2), we consider the inequality

$$I_3(P) \le I_{31}(P) + I_{32}(P), \tag{3.12}$$

where

$$I_{31}(P) = M\int_{S_n(\Omega;(\frac{4}{5}r,R))} \frac{-u(Q)\varphi(\Theta)}{t^{n-1}}\frac{\partial\varphi(\Phi)}{\partial n_\Phi}\,d\sigma_Q$$

and

$$I_{32}(P) = Mr\varphi(\Theta)\int_{S_n(\Omega;(\frac{4}{5}r,R))} \frac{-u(Q)r\varphi(\Theta)}{|P-Q|^n}\frac{\partial\varphi(\Phi)}{\partial n_\Phi}\,d\sigma_Q.$$

We first have

$$I_{31}(P) \le M\varphi(\Theta)r^{1-n-\aleph^-}\int_{S_n(\Omega;(\frac{4}{5}r,R))} -u(Q)t^{\aleph^-}\frac{\partial\varphi(\Phi)}{\partial n_\Phi}\,d\sigma_Q$$

$$\le MK\left(\frac{N+1}{N}R\right)^{\rho(\frac{N+1}{N}R)}\varphi(\Theta) \tag{3.13}$$

from (3.7). Next, we shall estimate $I_{32}(P)$. Take a sufficiently small positive number k such that $S_n(\Omega;(\frac{4}{5}r,R)) \subset B(P,\frac{1}{2}r)$ for any $P=(r,\Theta)\in\Pi(k)$, where

$$\Pi(k) = \left\{P=(r,\Theta)\in C_n(\Omega);\ \inf_{(1,z)\in\partial\Omega}\left|(1,\Theta)-(1,z)\right| < k, 0<r<\infty\right\},$$

and divide $C_n(\Omega)$ into two sets $\Pi(k)$ and $C_n(\Omega)-\Pi(k)$.

If $P=(r,\Theta)\in C_n(\Omega)-\Pi(k)$, then there exists a positive k' such that $|P-Q|\ge k'r$ for any $Q\in S_n(\Omega)$, and hence

$$I_{32}(P) \le M\int_{S_n(\Omega;(\frac{4}{5}r,R))} \frac{-u(Q)\varphi(\Theta)}{t^{n-1}}\frac{\partial\varphi(\Phi)}{\partial n_\Phi}\,d\sigma_Q$$

$$\le MK\left(\frac{N+1}{N}R\right)^{\rho(\frac{N+1}{N}R)}\varphi(\Theta), \tag{3.14}$$

which is similar to the estimate of $I_{31}(P)$.

We shall consider the case $P=(r,\Theta)\in\Pi(k)$. Now put

$$H_i(P) = \left\{Q\in S_n\left(\Omega;\left(\frac{4}{5}r,R\right)\right); 2^{i-1}\delta(P)\le|P-Q|<2^i\delta(P)\right\},$$

where $\delta(P) = \inf_{Q\in\partial C_n(\Omega)}|P-Q|$.

Since $S_n(\Omega) \cap \{Q \in \mathbf{R}^n : |P-Q| < \delta(P)\} = \varnothing$, we have

$$I_{32}(P) = M \sum_{i=1}^{i(P)} \int_{H_i(P)} \frac{-u(Q) r\varphi(\Theta)}{|P-Q|^n} \frac{\partial \varphi(\Phi)}{\partial n_\Phi} \, d\sigma_Q,$$

where $i(P)$ is a positive integer satisfying $2^{i(P)-1}\delta(P) \leq \frac{r}{2} < 2^{i(P)}\delta(P)$.

Since $r\varphi(\Theta) \leq M\delta(P)$ ($P = (r,\Theta) \in C_n(\Omega)$), similar to the estimate of $I_{31}(P)$ we obtain

$$\begin{aligned}
&\int_{H_i(P)} \frac{-u(Q) r\varphi(\Theta)}{|P-Q|^n} \frac{\partial \varphi(\Phi)}{\partial n_\Phi} \, d\sigma_Q \\
&\quad \leq \int_{H_i(P)} r\varphi(\Theta) \frac{-u(Q)}{(2^{i-1}\delta(P))^n} \frac{\partial \varphi(\Phi)}{\partial n_\Phi} \, d\sigma_Q \\
&\quad \leq M 2^{(1-i)n} \varphi^{1-n}(\Theta) \int_{H_i(P)} t^{1-n} - u(Q) \frac{\partial \varphi(\Phi)}{\partial n_\Phi} \, d\sigma_Q \\
&\quad \leq MK \left(\frac{N+1}{N} R \right)^{\rho(\frac{N+1}{N}R)} \varphi^{1-n}(\Theta)
\end{aligned}$$

for $i = 0, 1, 2, \ldots, i(P)$.

So

$$I_{32}(P) \leq MK \left(\frac{N+1}{N} R \right)^{\rho(\frac{N+1}{N}R)} \varphi^{1-n}(\Theta). \tag{3.15}$$

From (3.12), (3.13), (3.14), and (3.15) we see that

$$I_3(P) \leq MK \left(\frac{N+1}{N} R \right)^{\rho(\frac{N+1}{N}R)} \varphi^{1-n}(\Theta). \tag{3.16}$$

On the other hand, we have from (2.3) and (3.5) that

$$\begin{aligned}
I_4(P) &\leq M r^{\aleph^+} \varphi(\Theta) \int_{S_n(\Omega;R)} \frac{-u(Q)\varphi}{R^{1-\aleph^-}} \, dS_R \\
&\leq MKR^{\rho(R)} \varphi(\Theta).
\end{aligned} \tag{3.17}$$

We thus obtain (3.10), (3.11), (3.16), and (3.17) that

$$-u(P) \leq MK \left(1 + \left(\frac{N+1}{N} R \right)^{\rho(\frac{N+1}{N}R)} \right) \varphi^{1-n}(\Theta). \tag{3.18}$$

Case 2. $P = (r,\Theta) \in C_n(\Omega; (\frac{4}{5}, \frac{5}{4}])$ and $R = \frac{5}{4}r$.

Equation (3.8) gives

$$-u(P) = I_1(P) + I_5(P) + I_4(P),$$

where $I_1(P)$ and $I_4(P)$ are defined in Case 1 and

$$I_5(P) = \int_{S_n(\Omega;(1,R))} \mathcal{PI}_\Omega(P,Q) - u(Q) \, d\sigma_Q.$$

Similar to the estimate of $I_3(P)$ in Case 1 we have

$$I_5(P) \leq MK\left(\frac{N+1}{N}R\right)^{\rho(\frac{N+1}{N}R)}\varphi^{1-n}(\Theta),$$

which together with (3.10) and (3.17) gives (3.18).

Case 3. $P = (r, \Theta) \in C_n(\Omega; (0, \frac{4}{5}])$.

It is evident from (1.4) that we have $-u \leq K$, which also gives (3.18).

From (3.18) we finally have

$$u(P) \geq -KM\left(1 + \left(\frac{N+1}{N}R\right)^{\rho(\frac{N+1}{N}R)}\right)\varphi^{1-n}\theta,$$

which is the conclusion of Theorem 1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by the corresponding author BY. SP and BY prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

Author details
[1]Institute of Economic and Social Development Research, Zhejiang University of Finance and Economics, Hangzhou, 310018, P.R. China. [2]Mathematics Institute, Roskilde University, Roskilde, 4000, Denmark.

Acknowledgements
This work was partially supported by NSF Grant DMS-0913205.

Received: 24 November 2014 Accepted: 6 March 2015 Published online: 25 March 2015

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